Article

Majorization and Karamata Inequality

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Note

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The Author always appriciates every Contribution to this content- Majorization and Karamata Inequality.

Best Regard,

Pham Kim Hung

Chapter 1

Theory of Majorization

The theory of majorization and convex functions is an important and difficult part of inequalities, with many nice and powerful applications. will discuss in this article is **Karamata** inequality; however, it's necessary to review first some basic properties of majorization.

Definition 1. *Given two sequences* $(a) = (a_1, a_2, ..., a_n)$ *and* $(b) = (b_1, b_2, ..., b_n)$ *(where* $a_i, b_i \in \mathbb{R}\ \forall i \in \{1, 2, ..., n\}$*). We say that the sequence* (a) *majorizes the sequence* (b)*, and write* $(a) \gg (b)$*, if the following conditions are fulfilled*

$$\begin{aligned}
&a_1 \geq a_2 \geq ... \geq a_n\ ;\\
&b_1 \geq b_2 \geq ... \geq b_n\ ;\\
&a_1 + a_2 + ... + a_n = b_1 + b_2 + ... + b_n\ ;\\
&a_1 + a_2 + ... + a_k \geq b_1 + b_2 + ... + b_k\ \forall k \in \{1, 2, ...n - 1\}\ .
\end{aligned}$$

Definition 2. *For an arbitrary sequence* $(a) = (a_1, a_2, ..., a_n)$*, we denote* (a^*)*, a permutation of elements of* (a) *which are arranged in increasing order:* $(a^*) = (a_{i_1}, a_{i_2}, ..., a_{i_n})$ *with* $a_{i_1} \geq a_{i_2} \geq ... \geq a_{i_n}$ *and* $\{i_1, i_2, ..., i_n\} = \{1, 2, ..., n\}$.

Here are some basic properties of sequences.

Proposition 1. *Let* $a_1, a_2, ..., a_n$ *be real numbers and* $a = \frac{1}{n}(a_1 + a_2 + ... + a_n)$*, then*

$$(a_1, a_2, ..., a_n)^* \gg (a, a, ..., a).$$

Proposition 2. *Suppose that* $a_1 \geq a_2 \geq ... \geq a_n$ *and* $\pi = (\pi_1, \pi_2, ...\pi_n)$ *is an arbitrary permutation of* $(1, 2, ..., n)$*, then we have*

$$(a_1, a_2, ..., a_n) \gg (a_{\pi(1)}, a_{\pi(2)}, ..., a_{\pi(n)}).$$

Proposition 3. *Let* $(a) = (a_1, a_2, ..., a_n)$ *and* $(b) = (b_1, b_2, ..., b_n)$ *be two sequences of real numbers. We have that* (a^*) *majorizes* (b) *if the following conditions are fulfilled*

$$b_1 \geq b_2 \geq ... \geq b_n \ ;$$
$$a_1 + a_2 + ... + a_n = b_1 + b_2 + ... + b_n \ ;$$
$$a_1 + a_2 + ... + a_k \geq b_1 + b_2 + ... + b_k \ \forall k \in \{1, 2, ..., n-1\} \ ;$$

These properties are quite obvious: they can be proved directly from the definition of Majorization. The following results, especially the Symmetric Mjorization Criterion, will be most important in what follows.

Proposition 4. *If* $x_1 \geq x_2 \geq ... \geq x_n$ *and* $y_1 \geq y_2 \geq ... \geq y_n$ *are positive real numbers such that* $x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n$ *and* $\dfrac{x_i}{x_j} \geq \dfrac{y_i}{y_j}$ $\forall i < j$, *then*

$$(x_1, x_2, ..., x_n) \gg (y_1, y_2, ..., y_n).$$

Proof. To prove this assertion, we will use induction. Because $\dfrac{x_i}{x_1} \leq \dfrac{y_i}{y_1}$ for all $i \in \{1, 2, ..., n\}$, we get that

$$\frac{x_1 + x_2 + ... + x_n}{x_1} \leq \frac{y_1 + y_2 + ... + y_n}{y_1} \Rightarrow x_1 \geq y_1.$$

Consider two sequences $(x_1 + x_2, x_3, ..., x_n)$ and $(y_1 + y_2, y_3, ..., y_n)$. By the inductive hypothesis, we get

$$(x_1 + x_2, x_3, ..., x_n) \gg (y_1 + y_2, y_3, ..., y_n).$$

Combining this with the result that $x_1 \geq y_1$, we have the conclusion immediately.

$$\nabla$$

Theorem 1 (Symmetric Majorization Criterion). *Suppose that* $(a) = (a_1, a_2, ..., a_n)$ *and* $(b) = (b_1, b_2, ..., b_n)$ *are two sequences of real numbers; then* $(a^*) \gg (b^*)$ *if and only if for all real numbers* x *we have*

$$|a_1 - x| + |a_2 - x| + ... + |a_n - x| \geq |b_1 - x| + |b_2 - x| + ... + |b_n - x|.$$

Proof. To prove this theorem, we need to prove the following.

(i). Necessary condition. Suppose that $(a^*) \gg (b^*)$, then we need to prove that for all real numbers x

$$|a_1 - x| + |a_2 - x| + ... + |a_n - x| \geq |b_1 - x| + |b_2 - x| + ... + |b_n - x| \quad (\star)$$

Notice that $(\star)$ is just a direct application of **Karamata** inequality to the convex function $f(x) = |x - a|$; however, we will prove algebraically.

WLOG, assume that $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$, then $(a) \gg (b)$ by hypothesis. Obviously, $(\star)$ is true if $x \geq b_1$ or $x \leq b_n$, because in these cases, we have

$$\text{RHS} = |b_1 + b_2 + ... + b_n - nx| = |a_1 + a_2 + ... + a_n - nx| \leq \text{LHS}.$$

Consider the case when there exists an integer $k \in \{1, 2, ..., n-1\}$ for which $b_k \geq x \geq b_{k+1}$. In this case, we can remove the absolute value signs of the right-hand expression of $(\star)$

$$|b_1 - x| + |b_2 - x| + ... + |b_k - x| = b_1 + b_2 + ... + b_k - kx \ ;$$
$$|b_{k+1} - x| + |b_{k+2} - x| + ... + |b_n - x| = (n-k)x - b_{k+1} - b_{k+2} - ... - b_n \ ;$$

Moreover, we also have that

$$\sum_{i=1}^{k} |a_i - x| \geq -kx + \sum_{i=1}^{k} a_i,$$

and similarly,

$$\sum_{i=k+1}^{n} |a_i - x| = \sum_{i=k+1}^{n} |x - a_i| \geq (n-k)x - \sum_{i=k+1}^{n} a_i.$$

Combining the two results and noticing that $\sum\limits_{i=1}^{k} a_i \geq \sum\limits_{i=1}^{k} b_i$ and $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} b_i$, we get

$$\sum_{i=1}^{n} |a_i - x| \geq (n-2k)x + \sum_{i=1}^{k} a_i - \sum_{i=k+1}^{n} a_i$$

$$= 2\sum_{i=1}^{k} a_i - \sum_{i=1}^{n} a_i + (n-2k)x \geq 2\sum_{i=1}^{k} b_i - \sum_{i=1}^{n} b_i + (n-2k)x = \sum_{i=1}^{n} |b_i - x|.$$

This last inequality asserts our desired result.

(ii). Sufficient condition. Suppose that the inequality

$$|a_1 - x| + |a_2 - x| + ... + |a_n - x| \geq |b_1 - x| + |b_2 - x| + ... + |b_n - x| \ (\star\star)$$

has been already true for every real number x. We have to prove that $(a^*) \gg (b^*)$.

Without loss of generality, we may assume that $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$. Because $(\star\star)$ is true for all $x \in \mathbb{R}$, if we choose $x \geq \max\{a_i, b_i\}_{i=1}^{n}$ then

$$\sum_{i=1}^{n} |a_i - x| = nx - \sum_{i=1}^{n} a_i \ ; \ \sum_{i=1}^{n} |b_i - x| = nx - \sum_{i=1}^{n} b_i \ ;$$

$$\Rightarrow a_1 + a_2 + ... + a_n \leq b_1 + b_2 + ... + b_n.$$

Similarly, if we choose $x \le \min\{a_i, b_i\}_{i=1}^n$, then

$$\sum_{i=1}^n |a_i - x| = -nx + \sum_{i=1}^n a_i \ ; \ \sum_{i=1}^n |b_i - x| = -nx + \sum_{i=1}^n b_i \ ;$$

$$\Rightarrow a_1 + a_2 + ... + a_n \ge b_1 + b_2 + ... + b_n.$$

From these results, we get that $a_1+a_2+...+a_n = b_1+b_2+...+b_n$. Now suppose that x is a real number in $[a_k, a_{k+1}]$, then we need to prove that $a_1+a_2+...+a_k \ge b_1+b_2+...+b_k$. Indeed, we can eliminate the absolute value signs on the left-hand expression of $(\star\star)$ as follows

$$|a_1 - x| + |a_2 - x| + ... + |a_k - x| = a_1 + a_2 + ... + a_k - kx \quad ;$$

$$|a_{k+1} - x| + |a_{k+2} - x| + ... + |a_n - x| = (n-k)x - a_{k+1} - a_{k+2} - ... - a_n \quad ;$$

$$\Rightarrow \sum_{i=1}^n |a_i - x| = (n-2k)x + 2\sum_{i=1}^k a_i - \sum_{i=1}^n a_i.$$

Considering the right-hand side expression of $(\star\star)$, we have

$$\sum_{i=1}^n |b_i - x| = \sum_{i=1}^k |b_i - x| + \sum_{i=k+1}^n |x - b_i|$$

$$\ge -kx + \sum_{i=1}^k b_i + (n-k)x - \sum_{i=k+1}^n |b_i| = (n-2k)x + 2\sum_{i=1}^k |b_i| - \sum_{i=1}^n |b_i|.$$

From these relations and $(\star\star)$, we conclude that

$$(n-2k)x + 2\sum_{i=1}^k a_i - \sum_{i=1}^n a_i \ge (n-2k)x + 2\sum_{i=1}^k |b_i| - \sum_{i=1}^n |b_i|$$

$$\Rightarrow a_1 + a_2 + ... + a_k \ge b_1 + b_2 + ... + b_k,$$

which is exactly the desired result. The proof is completed.

∇

The Symmetric Majorization Criterion asserts that when we examine the majorization of two sequences, it's enough to examine only one conditional inequality which includes a real variable x. This is important because if we use the normal method, there may too many cases to check.

The essential importance of majorization lies in the **Karamata** inequality, which will be discussed right now.

Chapter 2

Karamata Inequality

Karamata inequality is a strong application of convex functions to inequalities. As we have already known, the function f is called convex on $\mathbb{I}$ if and only if $af(x)+bf(y) \geq f(ax+by)$ for all $x, y \in \mathbb{I}$ and for all $a, b \in [0,1]$. Moreover, we also have that f is convex if $f''(x) \geq 0 \ \forall x \in \mathbb{I}$. In the following proof of **Karamata** inequality, we only consider a convex function f when $f''(x) \geq 0$ because this case mainly appears in Mathematical Contests. This proof is also a nice application of **Abel** formula.

Theorem 2 (Karamata inequality). *If (a) and (b) two numbers sequences for which $(a^*) \gg (b^*)$ and f is a convex function twice differentiable on $\mathbb{I}$ then*

$$f(a_1) + f(a_2) + ... + f(a_n) \geq f(b_1) + f(b_2) + ... + f(b_n).$$

Proof. WLOG, assume that $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$. The inductive hypothesis yields $(a) = (a^*) \gg (b^*) = (b)$. Notice that f is a twice differentiable function on $\mathbb{I}$ (that means $f''(x) \geq 0$), so by **Mean Value** theorem, we claim that

$$f(x) - f(y) \geq (x-y)f'(y) \ \forall x, y \in \mathbb{I}.$$

From this result, we also have $f(a_i) - f(b_i) \geq (a_i - b_i)f'(b_i) \ \forall i \in \{1, 2, ..., n\}$. Therefore

$$\sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f(b_i) = \sum_{i=1}^{n} \left(f\left(a_i\right) - f\left(b_i\right)\right) \geq \sum_{i=1}^{n} (a_i - b_i) f'(b_i)$$

$$= (a_1 - b_1)(f'(b_1) - f'(b_2)) + (a_1 + a_2 - b_1 - b_2)(f'(b_2) - f'(b_3)) + ... +$$

$$+ \left(\sum_{i=1}^{n-1} a_i - \sum_{i=1}^{n-1} b_i\right)\left(f'\left(b_{n-1}\right) - f'\left(b_n\right)\right) + \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right) f'(b_n) \geq 0$$

because for all $k \in \{1, 2, ..., n\}$ we have $f'(b_k) \geq f'(b_{k+1})$ and $\sum\limits_{i=1}^{k} a_i \geq \sum\limits_{i=1}^{k} b_i$.

Comment. 1. If f is a non-decreasing function, it is certain that the last condition $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} b_i$ can be replaced by the stronger one $\sum\limits_{i=1}^{n} a_i \geq \sum\limits_{i=1}^{n} b_i$.

2. A similar result for concave functions is that

> ★ *If $(a) \gg (b)$ are number arrays and f is a concave function twice differentiable then*
> $$f(a_1) + f(a_2) + ... + f(a_n) \leq f(b_1) + f(b_2) + ... + f(b_n).$$

3. If f is convex (that means $\alpha f(a) + \beta f(b) \geq f(\alpha a + \beta b)$ $\forall \alpha, \beta \geq 0, \alpha + \beta = 1$) but not twice differentiable ($f''(x)$ does not exist), **Karamata** inequality is still true. A detailed proof can be seen in the book **Inequalities** written by G.H Hardy, J.E Littewood and G.Polya.

∇

The following examples should give you a sense of how this inequality can be used.

Example 2.1. *If f is a convex function then*

$$f(a) + f(b) + f(c) + f\left(\frac{a+b+c}{3}\right) \geq \frac{4}{3}\left(f\left(\frac{a+b}{2}\right) + f\left(\frac{b+c}{2}\right) + f\left(\frac{c+a}{2}\right)\right).$$

(Popoviciu-Titu Andreescu inequality)

Solution. WLOG, suppose that $a \geq b \geq c$. Consider the following number sequences

$$(x) = (a, a, a, b, t, t, t, b, b, c, c, c) \quad ; \quad (y) = (\alpha, \alpha, \alpha, \alpha, \beta, \beta, \beta, \beta, \gamma, \gamma, \gamma, \gamma) \quad ;$$

where

$$t = \frac{a+b+c}{3} \quad , \quad \alpha = \frac{a+b}{2} \quad , \quad \beta = \frac{a+c}{2} \quad , \quad \gamma = \frac{b+c}{2} \quad .$$

Clearly, we have that (y) is a monotonic sequence. Moreover

$$a \geq \alpha, 3a + b \geq 4\alpha, 3a + b + t \geq 4\alpha + \beta, 3a + b + 3t \geq 4\alpha + 3\beta,$$

$$3a + 2b + 3t \geq 4\alpha + 4\beta, 3a + 3b + 3t \geq 4\alpha + 4\beta + \gamma,$$

$$3a + 3b + 3t + c \geq 4\alpha + 4\beta + 2\gamma, 3a + 3b + 3t + 3c \geq 4\alpha + 4\beta + 4\gamma.$$

Thus $(x^*) \gg (y)$ and therefore $(x^*) \gg (y^*)$. By **Karamata** inequality, we conclude

$$3\left(f(x) + f(y) + f(z) + f(t)\right) \geq 4\left(f(\alpha) + f(\beta) + f(\gamma)\right),$$

which is exactly the desired result. We are done.

∇

Example 2.2 (Jensen Inequality). *If f is a convex function then*

$$f(a_1) + f(a_2) + ... + f(a_n) \geq nf\left(\frac{a_1 + a_2 + ... + a_n}{n}\right).$$

Solution. We use property 1 of majorization. Suppose that $a_1 \ge a_2 \ge ... \ge a_n$, then we have $(a_1, a_2, ..., a_n) \gg (a, a, ..., a)$ with $a = \frac{1}{n}(a_1 + a_2 + ... + a_n)$. Our problem is directly deduced from **Karamata** inequality for these two sequences.

∇

Example 2.3. *Let a, b, c, x, y, z be six real numbers in $\mathbb{I}$ satisfying*

$$a + b + c = x + y + z, \max(a, b, c) \ge \max(x, y, z), \min(a, b, c) \le \min(x, y, z),$$

then for every convex function f on $\mathbb{I}$, we have

$$f(a) + f(b) + f(c) \ge f(x) + f(y) + f(z).$$

Solution. Assume that $x \ge y \ge z$. The assumption implies $(a, b, c)^* \gg (x, y, z)$ and the conclusion follows from **Karamata** inequality.

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Example 2.4. *Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that*

$$(1 + a_1)(1 + a_2)...(1 + a_n) \le \left(1 + \frac{a_1^2}{a_2}\right)\left(1 + \frac{a_2^2}{a_3}\right)...\left(1 + \frac{a_n^2}{a_1}\right).$$

Solution. Our inequality is equivalent to

$$\ln(1+a_1)+\ln(1+a_2)+...+\ln(1+a_n) \le \ln\left(1 + \frac{a_1^2}{a_2}\right)+\ln\left(1 + \frac{a_2^2}{a_3}\right)+...+\ln\left(1 + \frac{a_n^2}{a_1}\right).$$

Suppose that the number sequence $(b) = (b_1, b_2, ..., b_n)$ is a permutation of $(\ln a_1, \ln a_2, ..., \ln a_n)$ which was rearranged in decreasing order. We may assume that $b_i = \ln a_{k_i}$, where $(k_1, k_2, ..., k_n)$ is a permutation of $(1, 2, .., n)$. Therefore the number sequence $(c) = (2\ln a_1 - \ln a_2, 2\ln a_2 - \ln a_3, ..., 2\ln a_n - \ln a_1)$ can be rearranged into a new one as

$$(c') = (2\ln a_{k_1} - \ln a_{k_1+1}, 2\ln a_{k_2} - \ln a_{k_2+1}, ..., 2\ln a_{k_n} - \ln a_{k_n+1}).$$

Because the number sequence $(b) = (\ln a_{k_1}, \ln a_{k_2}, ..., \ln a_{k_n})$ is decreasing, we must have $(c')^* \gg (b)$. By **Karamata** inequality, we conclude that for all convex function x then

$$f(c_1) + f(c_2) + ... + f(c_n) \ge f(b_1) + f(b_2) + ... + f(b_n),$$

where $c_i = 2\ln a_{k_i} - \ln a_{k_i+1}$ and $b_i = \ln a_{k_i}$ for all $i \in \{1, 2, ..., n\}$. Choosing $f(x) = \ln(1 + e^x)$, we have the desired result.

Comment. 1. A different choice of $f(x)$ can make a different problem. For example, with the convex function $f(x) = \sqrt{1 + e^x}$, we get

$$\sqrt{1 + a_1} + \sqrt{1 + a_2} + ... + \sqrt{1 + a_n} \le \sqrt{1 + \frac{a_1^2}{a_2}} + \sqrt{1 + \frac{a_2^2}{a_3}} + ... + \sqrt{1 + \frac{a_n^2}{a_1}}.$$

2. By **Cauchy-Schwarz** inequality, we can solve this problem according to the following estimation

$$\left(1+\frac{a_1^2}{a_2}\right)(1+a_2) \geq (1+a_1)^2.$$

∇

Example 2.5. *Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that*

$$\frac{a_1^2}{a_2^2+...+a_n^2}+...+\frac{a_n^2}{a_1^2+...+a_{n-1}^2} \geq \frac{a_1}{a_2+...+a_n}+...+\frac{a_n}{a_1+...+a_{n-1}}.$$

Solution. For each $i \in \{1, 2, ..., n\}$, we denote

$$y_i = \frac{a_i}{a_1+a_2+...+a_n},\ x_i = \frac{a_i^2}{a_1^2+a_2^2+...+a_n^2}$$

then $x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n = 1$. We need to prove that

$$\sum_{i=1}^{n} \frac{x_i}{1-x_i} \geq \sum_{i=1}^{n} \frac{y_i}{1-y_i}.$$

WLOG, assume that $a_1 \geq a_2 \geq ... \geq a_n$, then certainly $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$. Moreover, for all $i \geq j$, we also have

$$\frac{x_i}{x_j} = \frac{a_i^2}{a_j^2} \geq \frac{a_i}{a_j} = \frac{y_i}{y_j}.$$

By property 4, we deduce that $(x_1, x_2, ..., x_n) \gg (y_1, y_2, ..., y_n)$. Furthermore,

$$f(x) = \frac{x}{1-x}$$

is a convex function, so by **Karamata** inequality, the final result follows immediately.

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Example 2.6. *Suppose that $(a_1, a_2, ..., a_{2n})$ is a permutation of $(b_1, b_2, ..., b_{2n})$ which satisfies $b_1 \geq b_2 \geq ... \geq b_{2n} \geq 0$. Prove that*

$$(1+a_1a_2)(1+a_3a_4)...(1+a_{2n-1}a_{2n})$$

$$\leq (1+b_1b_2)(1+b_3b_4)...(1+b_{2n-1}b_{2n}).$$

Solution. Denote $f(x) = \ln(1+e^x)$ and $x_i = \ln a_i, y_i = \ln b_i$. We need to prove that

$$f(x_1+x_2)+f(x_3+x_4)+...+f(x_{2n-1}+x_{2n})$$

$$\leq f(y_1+y_2)+f(y_3+y_4)+...+f(y_{2n-1}+y_{2n}).$$

Consider the number sequences $(x) = (x_1 + x_2, x_3 + x_4, ..., x_{2n-1} + x_{2n})$ and $(y) = (y_1 + y_2, y_3 + y_4, ..., y_{2n-1} + y_{2n})$. Because $y_1 \geq y_2 \geq ... \geq y_n$, if $(x^*) = (x_1^*, x_2^*, ..., x_n^*)$ is a permutation of elements of (x) which are rearranged in the decreasing order, then

$$y_1 + y_2 + ... + y_{2k} \geq x_1^* + x_2^* + ... + x_{2k}^*,$$

and therefore $(y) \gg (x^*)$. The conclusion follows from **Karamata** inequality with the convex function $f(x)$ and two numbers sequences $(y) \gg (x^*)$.

∇

If these examples are just the beginner's applications of **Karamata** inequality, you will see much more clearly how effective this theorem is in combination with the Symmetric Majorization Criterion. Famous Turkevici's inequality is such an instance.

Example 2.7. *Let a, b, c, d be non-negative real numbers. Prove that*

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2.$$

(Turkevici's inequality)

SOLUTION. To prove this problem, we use the following lemma

★ *For all real numbers x, y, z, t then*

$$2(|x|+|y|+|z|+|t|)+|x+y+z+t| \geq |x+y|+|y+z|+|z+t|+|t+x|+|x+z|+|y+t|.$$

We will not give a detailed proof of this lemma now (because the next problem shows a nice generalization of this one, with a meticulous solution). At this time, we will clarify that this lemma, in combination with **Karamata** inequality, can directly give Turkevici's inequality. Indeed, let $a = e^{a_1}, b = e^{b_1}, c = e^{c_1}$ and $d = e^{d_1}$, our problem is

$$\sum_{cyc} e^{4a_1} + 2e^{a_1+b_1+c_1+d_1} \geq \sum_{sym} e^{2a_1+2b_1}.$$

Because $f(x) = e^x$ is convex, it suffices to prove that (a^*) majorizes (b^*) with

$$(a) = (4a_1, 4b_1, 4c_1, 4d_1, a_1 + b_1 + c_1 + d_1, a_1 + b_1 + c_1 + d_1) \ ;$$

$$(b) = (2a_1 + 2b_1, 2b_1 + 2c_1, 2c_1 + 2d_1, 2d_1 + 2a_1, 2a_1 + 2c_1, 2b_1 + 2d_1) \ ;$$

By the symmetric majorization criterion, we need to prove that for all $x_1 \in \mathbb{R}$ then

$$2|a_1 + b_1 + c_1 + d_1 - 4x_1| + \sum_{cyc} |4a_1 - 4x_1| \geq \sum_{sym} |2a_1 + 2b_1 - 4x_1|.$$

Letting now $x = a_1 - x_1, y = b_1 - x_1, z = c_1 - x_1, t = d_1 - x_1$, we obtain an equivalent form as

$$2\sum_{cyc} |x| + |\sum_{cyc} x| \geq \sum_{sym} |x + y|,$$

which is exactly the lemma shown above. We are done.

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Example 2.8. *Let $a_1, a_2, ..., a_n$ be non-negative real numbers. Prove that*

$$(n-1)(a_1^2 + a_2^2 + ... + a_n^2) + n\sqrt[n]{a_1^2 a_2^2 ... a_n^2} \geq (a_1 + a_2 + ... + a_n)^2.$$

Solution. We realize that Turkevici's inequality is a particular case of this general problem (for $n = 4$, it becomes Turkevici's). By using the same reasoning as in the preceding problem, we only need to prove that for all real numbers $x_1, x_2, ..., x_n$ then $(a^*) \gg (b^*)$ with

$$(a) = (\underbrace{2x_1, 2x_1, ..., 2x_1}_{n-1}, \underbrace{2x_2, 2x_2, ..., 2x_2}_{n-1}, ..., \underbrace{2x_n, 2x_n, ..., 2x_n}_{n-1}, \underbrace{2x, 2x, ..., 2x}_{n}) \ ;$$

$$(b) = (x_1 + x_1, x_1 + x_2, x_1 + x_3, ..., x_1 + x_n, x_2 + x_1, x_2 + x_2, ..., x_2 + x_n, ..., x_n + x_n) \ ;$$

and $x = \dfrac{1}{n}(x_1 + x_2 + ... + x_n)$. By the Symmetric Majorization Criterion, it suffices to prove that

$$(n-2)\sum_{i=1}^{n} |x_i| + |\sum_{i=1}^{n} x_i| \geq \sum_{i \neq j}^{n} |x_i + x_j|.$$

Denote $A = \{i \mid x_i \geq 0\}, B = \{i \mid x_i < 0\}$ and suppose that $|A| = m, |B| = k = n - m$. We will prove an equivalent form as follows: if $x_i \geq 0 \ \forall i \in \{1, 2, ..., n\}$ then

$$(n-2)\sum_{i \in A,B} x_i + |\sum_{i \in A} x_i - \sum_{j \in B} x_j| \geq \sum_{(i,j) \in A,B} (x_i + x_j) + \sum_{i \in A, j \in B} |x_i - x_j|.$$

Because $k + m = n$, we can rewrite the inequality above into

$$(k-1)\sum_{i \in A} x_i + (m-1)\sum_{j \in B} x_j + |\sum_{i \in A} x_i - \sum_{j \in B} x_j| \geq \sum_{i \in A, j \in B} |x_i - x_j| \ (\star)$$

Without loss of generality, we may assume that $\sum\limits_{i \in A} x_i \geq \sum\limits_{j \in B} x_j$. For each $i \in A$, let $|A_i| = \{j \in B | x_i \leq x_j\}$ and $r_i = |A_i|$. For each $j \in B$, let $|B_j| = \{i \in A | x_j \leq x_i\}$ and $s_j = |B_j|$. Thus the left-hand side expression in $(\star)$ can be rewritten as

$$\sum_{i \in A}(k - 2r_i)x_i + \sum_{j \in B}(m - 2s_j)x_j.$$

Therefore $(\star)$ becomes

$$\sum_{i \in A}(2r_i - 1)x_i + \sum_{j \in B}(2s_j - 1)x_j + |\sum_{i \in A} x_i - \sum_{j \in B} x_j| \geq 0$$

$$\Leftrightarrow \ \sum_{i \in A} r_i x_i + \sum_{j \in B}(s_j - 1)x_j \geq 0.$$

Notice that if $s_j \geq 1$ for all $j \in \{1, 2, ..., n\}$ then we have the desired result immediately. Otherwise, assume that there exists a number $s_l = 0$, then

$$\max_{i \in A \cup B} x_i \in B \;\Rightarrow r_i \geq 1 \;\forall i \in \{1, 2, ..., m\}.$$

Thus

$$\sum_{i \in A} r_i x_i + \sum_{j \in B} (s_j - 1) x_j \geq \sum_{i \in A} x_i - \sum_{j \in B} x_j \geq 0.$$

This problem is completely solved. The equality holds for $a_1 = a_2 = ... = a_n$ and $a_1 = a_2 = ... = a_{n-1}, a_n = 0$ up to permutation.

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Example 2.9. *Let $a_1, a_2, ..., a_n$ be positive real numbers with product 1. Prove that*

$$a_1 + a_2 + ... + a_n + n(n-2) \geq (n-1)\left(\frac{1}{\sqrt[n-1]{a_1}} + \frac{1}{\sqrt[n-1]{a_2}} + ... + \frac{1}{\sqrt[n-1]{a_n}}\right).$$

Solution. The inequality can be rewritten in the form

$$\sum_{i=1}^{n} a_i + n(n-2)\sqrt[n]{\prod_{i=1}^{n} a_i} \geq (n-1)\sum_{i=1}^{n} \sqrt[n-1]{\prod_{j \neq i} a_j}.$$

First we will prove the following result (that helps us prove the previous inequality immediately): if $x_1, x_2, ..., x_n$ are real numbers then $(\alpha^*) \gg (\beta^*)$ with

$$(\alpha) = (x_1, x_2, ..., x_n, x, x, ..., x) \;\; ;$$

$$(\beta) = (y_1, y_1, ..., y_1, y_2, y_2, ..., y_2, ..., y_n, y_n, ..., y_n) \;\; ;$$

where $x = \dfrac{1}{n}(x_1 + x_2 + ... + x_n)$, (α) includes $n(n-2)$ numbers x, (β) includes $n-1$ numbers y_k $(\forall k \in \{1, 2, ..., n\})$, and each number b_k is determined from $b_k = \dfrac{nx - x_i}{n-1}$.

Indeed, by the symmetric majorization criterion, we only need to prove that

$$|x_1| + |x_2| + ... + |x_n| + (n-2)|S| \geq |S - x_1| + |S - x_2| + ... + |S - x_n| \; (\star)$$

where $S = x_1 + x_2 + ... + x_n = nx$. In case $n = 3$, this becomes a well-known result

$$|x| + |y| + |z| + |x + y + z| \geq |x + y| + |y + z| + |z + x|.$$

In the general case, assume that $x_1 \geq x_2 \geq ... \geq x_n$. If $x_i \geq S \;\forall i \in \{1, 2, ..., n\}$ then

$$\text{RHS} = \sum_{i=1}^{n} (x_i - S) = -(n-1)S \leq (n-1)|S| \leq \sum_{i=1}^{n} |x_i| + (n-2)|S| = \text{LHS}.$$

and the conclusion follows. Case $x_i \le S\ \forall i \in \{1,2,...,n\}$ is proved similarly. We consider the final case. There exists an integer k $(1 \le k \le n-1)$ such that $x_k \ge S \ge x_{k+1}$. In this case, we can prove $(\star)$ simply as follows

$$\text{RHS} = \sum_{i=1}^{k}(x_i - S) + \sum_{i=k+1}^{n}(S - x_i) = \sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_{k+1} + (n-2k)S,$$

$$\le \sum_{i=1}^{n} |x_i| + (n-2k)|S| \le \sum_{i=1}^{n} |x_i| + (n-2)|S| = \text{LHS},$$

which is also the desired result. The problem is completely solved.

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Example 2.10. *Let $a_1, a_2, ..., a_n$ be non-negative real numbers. Prove that*

$$(n-1)\left(a_1^n + a_2^n + ... + a_n^n\right) + n a_1 a_2 ... a_n \ge (a_1 + a_2 + ... + a_n)\left(a_1^{n-1} + a_2^{n-1} + ... + a_n^{n-1}\right).$$

(Suranji's inequality)

Solution. We will prove first the following result for all real numbers $x_1, x_2, ..., x_n$

$$n(n-1)\sum_{i=1}^{n} |x_i| + n|S| \ge \sum_{i,j=1}^{n} |x_i + (n-1)x_j| \ (1)$$

in which $S = x_1 + x_2 + ... + x_n$. Indeed, let $z_i = |x_i|\ \forall i \in \{1,2,...,n\}$ and $A = \{i \mid 1 \le i \le n, i \in \mathbb{N}, x_i \ge 0\}$, $B = \{i \mid 1 \le i \le n, i \in \mathbb{N}, x_i < 0\}$. WLOG, we may assume that $A = \{1,2,...,k\}$ and $B = \{k+1, k+2, ..., n\}$, then $|A| = k, |B| = n-k = m$ and $z_i \ge 0$ for all $i \in A \cup B$. The inequality above becomes

$$n(n-1)\left(\sum_{i\in A} z_i + \sum_{j\in B} z_j\right) + n\left|\sum_{i\in A} z_i - \sum_{j\in B} z_j\right|$$

$$\ge \sum_{i,i'\in A} |z_i + (n-1)z_{i'}| + \sum_{j,j'\in B} |(n-1)z_j + z_{j'}| + \sum_{i\in A, j\in B}\left(|z_i - (n-1)z_j| + |(n-1)z_i - z_j|\right)$$

Because $n = k+m$, the previous inequality is equivalent to

$$n(m-1)\sum_{i\in A} z_i + n(k-1)\sum_{j\in B} z_j + n\left|\sum_{i\in A} z_i - \sum_{j\in B} z_j\right|$$

$$\ge \sum_{i\in A, j\in B} |z_i - (n-1)z_j| + \sum_{i\in A, j\in B} |(n-1)z_i - z_j| \quad (\star)$$

For each $i \in A$ we denote

$$B_i = \{\ j \in B \mid (n-1)z_i \ge z_j\ \}\ ;\ B_i' = \{j \in B \mid z_i \ge (n-1)z_j\}\ ;$$

For each $j \in B$ we denote

$$A_j = \{\ i \in A \big| (n-1)z_j \geq z_i\ \}\ ;\ A'_j = \{i \in A \big| z_j \geq (n-1)z_i\}\ ;$$

We have of course $B'_i \subset B_i \subset B$ and $A'_i \subset A_i \subset A$. After giving up the absolute value signs, the right-hand side expression of $(\star)$ is indeed equal to

$$\sum_{i\in A} (mn - 2|B'_i| - 2(n-1)|B_i|)\, z_i + \sum_{j\in B} \big(kn - 2|A'_j| - 2(n-1)|A_j|\big)\, z_j.$$

WLOG, we may assume that $\sum\limits_{i\in A} z_i \geq \sum\limits_{j\in B} z_j$. The inequality above becomes

$$\sum_{i\in A} (|B'_i| + (n-1)|B_i|)\, z_i + \sum_{j\in B} \big(|A'_j| + (n-1)|A_j| - n\big)\, z_j \geq 0.$$

Notice that if for all $j \in B$, we have $|A'_j| \geq 1$, then the conclusion follows immediately (because $A'_j \subset A_j$, then $|A_j| \geq 1$ and $|A'_j| + (n-1)|A_j| - n \geq 0\ \forall j \in B$). If not, we may assume that there exists a certain number $r \in B$ for which $|A'_r| = 0$, and therefore $|A_r| = 0$. Because $|A_r| = 0$, it follows that $(n-1)z_r \leq z_i$ for all $i \in A$. This implies that $|B_i| \geq |B'_i| \geq 1$ for all $i \in A$, therefore $|B'_i| + (n-1)|B_i| \geq n$ and we conclude that

$$\sum_{i\in A} (|B'_i| + (n-1)|B_i|)\, z_i + \sum_{j\in B} \big(|A'_j| + (n-1)|A_j| - n\big)\, z_j \geq n\sum_{i\in A} z_i - n\sum_{j\in B} z_j \geq 0.$$

Therefore (1) has been successfully proved and therefore Suranji's inequality follows immediately from **Karamata** inequality and the Symmetric Majorization Criterion.

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Example 2.11. *Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_1 \geq a_2 \geq ... \geq a_n$. Prove the following inequality*

$$\frac{a_1+a_2}{2} \cdot \frac{a_2+a_3}{2} \cdots \frac{a_n+a_1}{2} \leq \frac{a_1+a_2+a_3}{3} \cdot \frac{a_2+a_3+a_4}{3} \cdots \frac{a_n+a_1+a_2}{3}.$$

(V. Adya Asuren)

Solution. By using **Karamata** inequality for the concave function $f(x) = \ln x$, we only need to prove that the number sequence (x^*) majorizes the number sequence (y^*) in which $(x) = (x_1, x_2, ..., x_n)$, $(y) = (y_1, y_2, ..., y_n)$ and for each $i \in \{1, 2, ..., n\}$

$$x_i = \frac{a_i + a_{i+1}}{2},\ y_i = \frac{a_i + a_{i+1} + a_{i+2}}{3}$$

(with the common notation $a_{n+1} = a_1$ and $a_{n+2} = a_2$). According to the Symmetric Majorization Criterion, it suffices to prove the following inequality

$$3\left(\sum_{i=1}^{n} |z_i + z_{i+1}|\right) \geq 2\left(\sum_{i=1}^{n} |z_i + z_{i+1} + z_{i+2}|\right)\ (\star)$$

for all real numbers $z_1 \geq z_2 \geq \dots \geq z_n$ and $z_{n+1},\ z_{n+2}$ stand for $z_1,\ z_2$ respectively.

Notice that $(*)$ is obviously true if $z_i \geq 0$ for all $i = 1, 2, \dots, n$. Otherwise, assume that $z_1 \geq z_2 \geq \dots \geq z_k \geq 0 > z_{k+1} \geq \dots \geq z_n$. We realize first that it's enough to consider $(\star)$ for 8 numbers (instead of n numbers). Now consider it for 8 numbers $z_1, z_2, \dots, z_8$. For each number $i \in \{1, 2, \dots, 8\}$, we denote $c_i = |z_i|$, then $c_i \geq 0$. To prove this problem, we will prove first the most difficult case $z_1 \geq z_2 \geq z_3 \geq z_4 \geq 0 \geq z_5 \geq z_6 \geq z_7 \geq z_8$. Giving up the absolute value signs, the problem becomes

$$3(c_1 + 2c_2 + 2c_3 + c_4 + c_5 + 2c_6 + 2c_7 + c_8 + |c_4 - c_5| + |c_8 - c_1|)$$

$$\geq 2(c_1+2c_2+2c_3+c_4+|c_3+c_4-c_5|+|c_4-c_5-c_6|+c_5+2c_6+2c_7+c_8+|c_7+c_8-c_1|+|c_8-c_1-c_2|)$$

$$\Leftrightarrow\ c_1 + 2c_2 + 2c_3 + c_4 + c_5 + 2c_6 + 2c_7 + c_8 + 3|c_4 - c_5| + 3|c_8 - c_1|$$

$$\geq 2|c_3 + c_4 - c_5| + 2|c_4 - c_5 - c_6| + 2|c_7 + c_8 - c_1| + 2|c_8 - c_1 - c_2|$$

Clearly, this inequality is obtained by adding the following results

$$\begin{aligned} 2|c_4 - c_5| + 2c_3 &\geq 2|c_3 + c_4 + c_5| \\ 2|c_8 - c_1| + 2c_7 &\geq 2|c_7 + c_8 - c_1| \\ |c_4 - c_5| + c_4 + c_5 + 2c_6 &\geq 2|c_4 - c_5 - c_6| \\ |c_8 - c_1| + c_8 + c_1 + 2c_2 &\geq 2|c_8 - c_1 - c_2| \end{aligned}$$

For other cases when there exist exactly three (or five); two (or six); only one (or seven) non-negative numbers in $\{z_1, z_2, \dots, z_8\}$, the problem is proved completely similarly (indeed, notice that, for example, if $z_1 \geq z_2 \geq z_3 \geq 0 \geq z_4 \geq z_5 \geq z_6 \geq z_7 \geq z_8$ then we only need to consider the similar but simpler inequality of seven numbers after eliminating z_6). Therefore $(\star)$ is proved and the conclusion follows immediately.

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Using **Karamata** inequality together with the theory of majorization like we have just done it is an original method for algebraic inequalities. By this method, a purely algebraic problem can be transformed to a linear inequality with absolute signs, which is essentially an arithmetic problem, and which can have many original solutions.